

CONVERGENCE OF DYNAMICS AND THE PERRON-FROBENIUS OPERATOR

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ABSTRACT. We complete the picture how the asymptotic behavior of a dynamical system is reflected by properties of the associated Perron-Frobenius operator. Our main result states that strong convergence of the powers of the Perron-Frobenius operator is equivalent to setwise convergence of the underlying dynamic in the measure algebra. This situation is furthermore characterized by uniform mixing-like properties of the system.

1. INTRODUCTION AND PRELIMINARIES

Let $(\Omega, \Sigma, \mu; \varphi)$ be a measure preserving dynamical system, i.e. a probability space (Ω, Σ, μ) endowed with a measurable transformation $\varphi: \Omega \rightarrow \Omega$ such that the push-forward measure $\varphi_*\mu$ equals μ . The so-called *Perron-Frobenius operator* $P \in \mathcal{L}(L^1(\Omega, \Sigma, \mu))$ can be defined by the Radon-Nikodym theorem as the unique linear and positive operator satisfying

$$(1.1) \quad \int_A P f \, d\mu = \int_{\varphi^{-1}(A)} f \, d\mu$$

for all $f \in L^1(\Omega, \Sigma, \mu)$ and $A \in \Sigma$. One easily checks that its adjoint P^* is the associated *Koopman operator* $T \in \mathcal{L}(L^\infty(\Omega, \Sigma, \mu))$ given by $Tf := f \circ \varphi$, see [3, Prop 4.3.1]. In particular, P is a bi-Markov operator, i.e. P is positive, $P\mathbb{1} = \mathbb{1}$ and $P^*\mathbb{1} = \mathbb{1}$. Since the system is measure preserving, T actually operates on every $L^p(\Omega, \Sigma, \mu)$ -space and P can also be obtained as the adjoint of T defined on $L^2(\Omega, \Sigma, \mu)$, extended to $L^1(\Omega, \Sigma, \mu)$.

It is well-known that some mixing properties of a dynamical system are described by the asymptotic behavior of the powers of the associated Perron-Frobenius operator. We recall two examples from [3, Thm 4.4.2]: Firstly, the measure preserving dynamical system $(\Omega, \Sigma, \mu; \varphi)$ is *ergodic*, meaning that the measure of each set $A \in \Sigma$ such that $\varphi^{-1}(A)$ equals A up to a nullset is either 0 or 1, if and only if the fixed space of P consists of constant functions only. Since P is a Dunford-Schwartz operator and therefore mean ergodic [5, Thm 8.24], ergodicity of the system is furthermore characterized by strong convergence of the Cesàro averages of P to $\mathbb{1} \otimes \mathbb{1}$, the bi-Markov projection onto the constant functions. Secondly, the system is *(strongly) mixing*, i.e.

$$(1.2) \quad \lim_{n \rightarrow \infty} \mu(\varphi^{-n}(A) \cap B) = \mu(A)\mu(B) \text{ for all } A, B \in \Sigma,$$

if and only if the sequence of powers $(P^n)_{n \in \mathbb{N}}$ converges in the weak operator topology to the just mentioned projection $\mathbb{1} \otimes \mathbb{1}$.

In this article we are concerned with *strong convergence* of the sequence $(P^n)_{n \in \mathbb{N}}$, i.e. convergence with respect to the strong operator topology, and its relation to

Date: March 30, 2017.

2010 Mathematics Subject Classification. Primary: 37A05, Secondary: 28D05, 37A25, 47A35.

Key words and phrases. Perron-Frobenius operator, exactness, mixing, convergence.

The author thanks Jochen Glück for a fruitful discussion on the results of Section 2 during his stay at the University of Potsdam in August 2016.

the asymptotic behavior of the underlying measure preserving dynamical system. It seems that this situation has never yet been characterized by a mixing-like property analogously to (1.2). Such a characterization is provided in Section 2: while it is easy to see that strong convergence of $(P^n)_{n \in \mathbb{N}}$ to the rank-one projection $\mathbf{1} \otimes \mathbf{1}$ is equivalent to the fact that the convergence in (1.2) is uniform in $A \in \Sigma$, we furthermore show that this is in turn equivalent to the formally weaker property that for any $B \in \Sigma$ there exists $D \in \Sigma$ with $\mu(D) > 0$ such that

$$(1.3) \quad \lim_{n \rightarrow \infty} \sup_{A \in \Sigma, A \subseteq D} |\mu(\varphi^{-n}(A) \cap B) - \mu(A)\mu(B)| = 0.$$

Moreover, this property is also related to exactness of the system, which is defined as follows. For each $n \in \mathbb{N}$ set $\Sigma_n := \{\varphi^{-n}(A) : A \in \Sigma\}$. We define the sub- σ -algebra $\Sigma_\infty := \bigcap_{n \in \mathbb{N}} \Sigma_n$ and denote its completion within Σ by $\overline{\Sigma_\infty}$. The measure preserving system $(\Omega, \Sigma, \mu; \varphi)$ is called *exact* if the measure of each element of Σ_∞ is either 0 or 1. It follows from [3, Thm 3.2.3 and 4.4.2] that in case where the measure preserving dynamical system is *bimeasurable*, i.e. $\varphi^{-1}(A), \varphi(A) \in \Sigma$ for every $A \in \Sigma$, the following assertions are equivalent:

- (i) $\lim \mu(\varphi^n(A)) = 1$ for each $A \in \Sigma$ with $\mu(A) > 0$.
- (ii) The sequence $(P^n)_{n \in \mathbb{N}}$ converges strongly to $\mathbf{1} \otimes \mathbf{1}$.
- (iii) The system is exact.

Note that the equivalence of (ii) and (iii) follows from [10][Cor 4.1]. In view of this characterization, we call the Frobenius-Perron operator P *exact* if assertion (ii) holds.

As a consequence of these equivalences we obtain that any bimeasurable exact system is mixing and thus ergodic. However, if the system $(\Omega, \Sigma, \mu; \varphi)$ is not ergodic, the powers $(P^n)_{n \in \mathbb{N}}$ can still converge to a projection onto the fixed space of P , which is no longer one-dimensional. In Section 3 we extend the above equivalences to non-ergodic systems by showing that in general strong convergence of $(P^n)_{n \in \mathbb{N}}$ is equivalent to setwise convergences of the iterates $(\varphi^n)_{n \in \mathbb{N}}$ in the measure algebra.

In order to make the last statement precise, we have to introduce a little more notation. On the σ -algebra Σ we define the equivalence relation

$$A \sim B :\Leftrightarrow \mu(A \triangle B) = \mu(A \setminus B) + \mu(B \setminus A) = 0$$

and denote by Σ/\sim the set of its equivalence classes. Note that Σ/\sim is called the *measure algebra*, which is a complete metric space with respect to the metric $d(A, B) := \mu(A \triangle B)$, cf. [1, Thm 1.12.6]. For the sake of simplicity we omit the distinction in notation between a set $A \in \Sigma$ and its equivalence class in Σ/\sim . Furthermore, we denote by

$$\Sigma_{\text{inv}} := \{A \in \Sigma : A = \varphi^{-1}(A)\}$$

the sub- σ -algebra of φ -invariant sets and by $\overline{\Sigma_{\text{inv}}}$ its completion within Σ . Then clearly $\Sigma_{\text{inv}} \subseteq \Sigma_\infty$ and therefore $\overline{\Sigma_{\text{inv}}} \subseteq \overline{\Sigma_\infty}$.

For any $A \subseteq \Omega$ we define the *minimal invariant superset* A^* as the intersection of all $B \subseteq \Omega$ containing A such that $\varphi^{-1}(B) = B$. If φ is bimeasurable, then the minimal invariant superset of any $A \in \Sigma$ is automatically measurable, see Lemma 3.5. Let us emphasize that our definition of A^* is different from that in the literature, where A^* is usually defined as the minimal element of Σ_{inv}/\sim that contains A up to a nullset, cf. [6, p. 21]. It is not difficult to see that both definitions coincide up to a nullset if and only if φ preserves nullsets.

Finally, let us denote by $\mathbb{E}(f|\Sigma_{\text{inv}})$ the conditional expectation of $f \in L^1(\Omega, \Sigma, \mu)$ with respect to Σ_{inv} , i.e. the unique element in $L^1(\Omega, \Sigma, \mu)$ that contains a Σ_{inv} -measurable representative and that

$$\int_B \mathbb{E}(f|\Sigma_{\text{inv}}) \, d\mu = \int_B f \, d\mu$$

for all $B \in \Sigma_{\text{inv}}$. It is also worth noting that $\mathbb{E}(f|\Sigma_{\text{inv}}) = \mathbb{E}(f|\overline{\Sigma_{\text{inv}}})$ in the sense of $L^1(\Omega, \Sigma, \mu)$.

Our main theorem now reads as follows.

Theorem 1.1. *Let $(\Omega, \Sigma, \mu; \varphi)$ be a bimeasurable measure preserving dynamical system $(\Omega, \Sigma, \mu; \varphi)$. Then the following assertions are equivalent.*

- (i) *For all $A \in \Sigma$ the sequence $(\varphi^n(A))_{n \in \mathbb{N}}$ converges in Σ/\sim .*
- (ii) *For all $f \in L^1(\Omega, \Sigma, \mu)$ the sequence $(P^n f)_{n \in \mathbb{N}}$ converges in $L^1(\Omega, \Sigma, \mu)$.*
- (iii) *$\overline{\Sigma_\infty} = \overline{\Sigma_{\text{inv}}}$.*

If these equivalent assertions hold, then the occurring limits can be identified as follows: for each $f \in L^1(\Omega, \Sigma, \mu)$ the sequence $(P^n f)_{n \in \mathbb{N}}$ converges to $\mathbb{E}(f|\Sigma_{\text{inv}})$ and for each $A \in \Sigma$ the sequence $(\varphi^n(A))_{n \in \mathbb{N}}$ converges in Σ/\sim to the minimal invariant superset A^ .*

Let us remark that the equivalence of (ii) and (iii) is not new, but is essentially contained in [10] and [2].

It is also worth mentioning that in Theorem 1.1 φ is not assumed to preserve nullsets, i.e. modifying A by a nullset will in general change the sequence $(\varphi^n(A))$ and its limit in Σ/\sim . Moreover, when starting with a transformation of the measure algebra $\varphi^*: \Sigma/\sim \rightarrow \Sigma/\sim$ that induces an exact Perron-Frobenius operator and possesses bimeasurable liftings $\varphi: \Omega \rightarrow \Omega$, Theorem 1.1 states that the convergence of $(\varphi^n(A))$ does not depend on the choice of the lifting, whereas its limit does. In order to illustrate this phenomenon consider the following simple example: Let $\Omega := \{1, 2, 3\}$ be endowed with the probability measure $\mu(1) := \mu(3) := 1/2$ and $\mu(2) := 0$. Then $\varphi(1) := 1$, $\varphi(2) := \varphi(3) := 3$ defines a bimeasurable measure preserving transformation $\varphi: \Omega \rightarrow \Omega$. For $A := \{1, 2\}$ we have $\varphi^n(A) = \{1, 3\}$ for every $n \in \mathbb{N}$ and therefore $\lim \varphi^n(A) = \{1, 3\}$ in Σ/\sim . Note that $\{1, 3\} \sim \Omega$ and Ω is in fact the minimal invariant superset of A with respect to the above definition. On the other hand, $A \sim \{1\}$ and $\varphi^n(\{1\}) = \{1\}$ for each $n \in \mathbb{N}$. However, $\{1\}$ is essentially different from Ω .

Let us conclude this introduction with a short note on bimeasurability. It was proven in [12], see also [11], that a Borel-measurable function $\varphi: X \rightarrow Y$ between two Polish spaces X and Y preserves Borel sets if and only if $\varphi^{-1}(\{y\})$ is at most countable for all but at most countably many $y \in Y$. For instance, the transformation $\varphi(x) := 2x \bmod 1$ is bimeasurable on $[0, 1]$ with respect to the Borel- σ -algebra. In some situations bimeasurability can be seen as a reasonable substitute for invertibility, cf. [13]. However, if φ is bimeasurable and bijective, then $P = T^{-1}$ and the powers of P can thus not be strongly convergent unless $P = I$. In fact, if P is strongly convergent, it follows from

$$\|f - Pf\| \leq \|T^n\| \cdot \|P^n f - P^{n+1} f\| \rightarrow 0 \quad (n \rightarrow \infty)$$

that $Pf = f$ for any f .

2. UNIFORM MIXING PROPERTIES

In the following we characterize strong convergence of the powers of the Perron-Frobenius operator P by mixing-like properties of the underlying measure preserving dynamical system $(\Omega, \Sigma, \mu; \varphi)$. In doing so, we make use of lower bound techniques originally invented by Lasota and Yorke in [9, 8] to provide a sufficient

condition for exactness of P . Ding extended this approach in [4] to characterize strong convergence of the sequence $(P^n)_{n \in \mathbb{N}}$ in general. He proved that the limit $\lim_{n \rightarrow \infty} P^n f$ exists for any $f \in L^1(\Omega, \Sigma, \mu)$ if and only if for each $0 < f \in L^1(\Omega, \Sigma, \mu)$ there exists some $0 < h \in L^1(\Omega, \Sigma, \mu)$ such that

$$\lim_{n \rightarrow \infty} \|(P^n f - h)^-\|_{L^1} = 0,$$

where we use the notation $g = g^+ - g^-$ for the positive and negative part of a function $g \in L^1$. This work has not yet gotten the attention it deserves and it acts as the key to the results in this section. We also refer to [7] for a detailed discussion and further generalizations of lower bound techniques.

We start with a characterization of strong convergence of $(P^n)_{n \in \mathbb{N}}$ in general.

Proposition 2.1. *Let $(\Omega, \Sigma, \mu; \varphi)$ be a measure preserving dynamical system with associated Perron-Frobenius operator $P \in \mathcal{L}(L^1(\Omega, \Sigma, \mu))$. Then the following assertions are equivalent:*

- (i) *The sequence $(P^n)_{n \in \mathbb{N}}$ converges with respect to the strong operator topology.*
- (ii) *For each $B \in \Sigma$ with $\mu(B) > 0$ there exists $D \in \Sigma$ with $\mu(D) > 0$ and a constant $c > 0$ such that*

$$\liminf_{n \rightarrow \infty} \inf_{A \in \Sigma} (\mu(\varphi^{-n}(A) \cap B) - c \cdot \mu(D \cap A)) \geq 0.$$

Proof. (i) \Rightarrow (ii): Let $(P^n)_{n \in \mathbb{N}}$ be strongly convergent and let $B \in \Sigma$ such that $\mu(B) > 0$. Then the limit $f_B := \lim_{n \rightarrow \infty} P^n \mathbf{1}_B$ is a positive function with $\|f_B\|_{L^1} = \mu(B) > 0$. Hence, there exists $D \in \Sigma$ with $\mu(D) > 0$ and a constant $c > 0$ such that $f_B \geq c \mathbf{1}_D$; in particular $\lim_{n \rightarrow \infty} \|(P^n \mathbf{1}_B - c \mathbf{1}_D)^-\|_{L^1} = 0$. Since

$$\begin{aligned} \inf_{A \in \Sigma} (\mu(\varphi^{-n}(A) \cap B) - c \cdot \mu(D \cap A)) &= \inf_{A \in \Sigma} \int_A (P^n \mathbf{1}_B - c \mathbf{1}_D) \, d\mu \\ &\geq \inf_{A \in \Sigma} \int_A (P^n \mathbf{1}_B - c \mathbf{1}_D)^+ \, d\mu - \int_\Omega (P^n \mathbf{1}_B - c \mathbf{1}_D)^- \, d\mu \end{aligned}$$

for each $n \in \mathbb{N}$, assertion (ii) follows.

(ii) \Rightarrow (i): Let $0 < f \in L^1(\Omega, \Sigma, \mu)$ and choose $B \in \Sigma$ with $\mu(B) > 0$ and $d > 0$ such that $f \geq d \mathbf{1}_B$. Now pick $D \in \Sigma$ and $c > 0$ according to assertion (ii). By (1.1), the definition of P , we obtain

$$\begin{aligned} \liminf_{n \rightarrow \infty} - \int_\Omega (P^n \mathbf{1}_B - c \mathbf{1}_D)^- \, d\mu &= \liminf_{n \rightarrow \infty} \inf_{A \in \Sigma} \int_A (P^n \mathbf{1}_B - c \mathbf{1}_D) \, d\mu \\ &= \liminf_{n \rightarrow \infty} \inf_{A \in \Sigma} (\mu(\varphi^{-n}(A) \cap B) - c \cdot \mu(D \cap A)) \geq 0 \end{aligned}$$

and hence that $\lim_{n \rightarrow \infty} \|(P^n \mathbf{1}_B - c \mathbf{1}_D)^-\|_{L^1} = 0$. Now it follows from

$$(P^n f - d c \mathbf{1}_D)^- \leq (P^n f - d P^n \mathbf{1}_B)^- + (d P^n \mathbf{1}_B - d c \mathbf{1}_D)^- = d (P^n \mathbf{1}_B - c \mathbf{1}_D)^-$$

that $\lim_{n \rightarrow \infty} \|(P^n f - d c \mathbf{1}_D)^-\|_{L^1} = 0$. Therefore, we obtain from Ding's theorem [4, Thm 1.1] that $(P^n)_{n \in \mathbb{N}}$ converges with respect to the strong operator topology. \square

Next we characterize exactness of P by a mixing-like properties of the dynamical system.

Theorem 2.2. *Let $(\Omega, \Sigma, \mu; \varphi)$ be a measure preserving dynamical system with associated Perron-Frobenius operator $P \in \mathcal{L}(L^1(\Omega, \Sigma, \mu))$. Then the following assertions are equivalent:*

- (i) *The operator P is exact, i.e. $\lim_{n \rightarrow \infty} P^n f = \int_\Omega f \, d\mu \cdot \mathbf{1}$ for each $f \in L^1(\Omega, \Sigma, \mu)$.*
- (ii) *For each $B \in \Sigma$*

$$\lim_{n \rightarrow \infty} \sup_{A \in \Sigma} |\mu(\varphi^{-n}(A) \cap B) - \mu(A)\mu(B)| = 0.$$

(iii) For each $B \in \Sigma$ there exists $D \in \Sigma$ with $\mu(D) > 0$ such that

$$\lim_{n \rightarrow \infty} \sup_{A \in \Sigma_D} |\mu(\varphi^{-n}(A) \cap B) - \mu(A)\mu(B)| = 0,$$

where $\Sigma_D := \{A \cap D : A \in \Sigma\}$ denotes the trace σ -algebra.

Proof. (i) \Rightarrow (ii): If P is exact, then it follows from

$$\begin{aligned} \sup_{A \in \Sigma} |\mu(\varphi^{-n}(A) \cap B) - \mu(A)\mu(B)| &= \sup_{A \in \Sigma} \left| \int_A (P^n \mathbf{1}_B - \mu(B)\mathbf{1}) \, d\mu \right| \\ &\leq \int_{\Omega} |P^n \mathbf{1}_B - \mu(B)\mathbf{1}| \, d\mu \end{aligned}$$

that assertion (ii) holds.

(ii) \Rightarrow (iii): trivial.

(iii) \Rightarrow (i): Let $B \in \Sigma$ be arbitrary and let $D \in \Sigma$ according to assertion (iii). Then for a given $\varepsilon > 0$ we find $N \in \mathbb{N}$ such that

$$|-\mu(\varphi^{-n}(A) \cap B) + \mu(A)\mu(B)| < \varepsilon \text{ for all } n \geq N \text{ and } A \in \Sigma, A \subseteq D.$$

In particular, we obtain that

$$\mu(\varphi^{-n}(A) \cap B) - \mu(A \cap D)\mu(B) > -\varepsilon \text{ for all } n \geq N \text{ and } A \in \Sigma.$$

Now it follows from Proposition 2.1 that $(P^n)_{n \in \mathbb{N}}$ converges with respect to the strong operator topology.

In order to prove that P is exact, it remains to show that the fixed space of P is one-dimensional and thus consists of constant functions only. In view of [3, Thm 4.4.2] this is equivalent to ergodicity of the system. Let $\Omega_1 \in \Sigma$ be an invariant set, i.e. $\varphi^{-1}(\Omega_1) \sim \Omega_1$, and note that $\Omega_2 := \Omega \setminus \Omega_1$ is also invariant by [5, Lem 6.17]. Define $B := \Omega_1$ and choose $D \in \Sigma$ with $\mu(D) > 0$ according to assertion (iii). First, we consider the case that the set $A := D \cap B$ has positive measure. Since $\varphi^{-1}(A)$ is contained in B up to a nullset, we obtain from assertion (iii) that

$$\mu(A) = \mu(\varphi^{-n}(A) \cap B) \rightarrow \mu(A)\mu(B) \text{ as } n \rightarrow \infty.$$

Therefore, $\mu(B) = 1$ and thus $\mu(\Omega_2) = 0$. Now assume that $D \cap B$ is a nullset. Since $A := D$ is contained in Ω_2 up to a nullset, we obtain from assertion (iii) that

$$\mu(\emptyset) = \mu(\varphi^{-n}(A) \cap B) \rightarrow \mu(A)\mu(B) \text{ as } n \rightarrow \infty.$$

Since $\mu(A) > 0$ this implies that $\mu(B) = 0$ and thus $\mu(\Omega_2) = 1$. In both cases we have that $\mu(\Omega_1) \in \{0, 1\}$ which proves that the system is ergodic. Therefore, $(P^n)_{n \in \mathbb{N}}$ converges in the strong operator topology to $\mathbf{1} \otimes \mathbf{1}$. \square

Let us conclude this section with a remark on bimeasurable systems. If the system is mixing, it was shown in [13, Thm 1] that

$$(2.1) \quad \lim_{n \rightarrow \infty} \mu(\varphi^n(A) \cap B) = \lim_{n \rightarrow \infty} \mu(\varphi^n(A))\mu(B)$$

for all $A, B \in \Sigma$. We show next that the system is exact if and only if the convergence in (2.1) holds uniformly in $B \in \Sigma$.

Lemma 2.3. *Let $(\Omega, \Sigma, \mu; \varphi)$ be a bimeasurable measure preserving dynamical system. Then $(\Omega, \Sigma, \mu; \varphi)$ is exact if and only if*

$$(2.2) \quad \lim_{n \rightarrow \infty} \sup_{B \in \Sigma} |\mu(\varphi^n(A) \cap B) - \lim_{m \rightarrow \infty} \mu(\varphi^m(A))\mu(B)| = 0$$

for all $A \in \Sigma$.

Proof. First note that for every $A \in \Sigma$ it follows from $\varphi^{-1}(\varphi(A)) \supseteq A$ that $\mu(\varphi(A)) = \mu(\varphi^{-1}(\varphi(A))) \geq \mu(A)$. Therefore, the limit of the monotone sequence $(\mu(\varphi^m(A)))_{m \in \mathbb{N}}$ always exists.

Now assume that the system is exact and let $A \in \Sigma$. If $\mu(\varphi^n(A)) = 0$ for all $n \in \mathbb{N}_0$, then assertion (2.2) is trivial. Otherwise, we have $\lim \mu(\varphi^n(A)) = 1$ by [3, Thm 3.2.3] and therefore

$$\sup_{B \in \Sigma} |\mu(\varphi^n(A) \cap B) - \mu(B)| = \sup_{B \in \Sigma} \mu(B \setminus \varphi^n(A)) \leq \mu(\Omega \setminus \varphi^n(A)) \rightarrow 0$$

as n tends to infinity.

Conversely, assume that (2.2) holds. Fix $A \in \Sigma$, set $a := \lim \mu(\varphi^m(A))$ and let $B_n := \Omega \setminus \varphi^n(A)$. Then

$$|\mu(\varphi^n(A) \cap B_n) - a \cdot \mu(B_n)| = a \cdot \mu(\Omega \setminus \varphi^n(A)) \rightarrow a(1 - a)$$

as n tends to infinity. Hence, either $a = 0$ and consequently A is a nullset by the preliminary note, or $a = 1$. By [3, Thm 3.2.3], this shows that the system is exact \square

3. CONVERGENCE OF DYNAMICS

In this section we prove our main result, Theorem 1.1, which is prepared by a series of lemmas and preliminary remarks.

Remark 3.1. Let us recall that the Cesàro averages of the Perron-Frobenius operator P corresponding to a measure preserving dynamical system $(\Omega, \Sigma, \mu; \varphi)$ always converge strongly to the conditional expectation $\mathbb{E}(\cdot | \Sigma_{\text{inv}}) = \mathbb{E}(\cdot | \Sigma_{\text{inv}})$, see e.g. [5, Rem 13.24]. This also implies that the Perron-Frobenius operator and the Koopman operator on $L^1(\Omega, \Sigma, \mu)$ share the same set of fixed points, namely $L^1(\Omega, \Sigma_{\text{inv}}, \mu)$.

Lemma 3.2. *Let $(\Omega, \Sigma, \mu; \varphi)$ be a measure preserving dynamical system with associated Perron-Frobenius operator $P \in \mathcal{L}(L^1(\Omega, \Sigma, \mu))$. For each $A \in \Sigma$ and $m \in \mathbb{N}_0$ we have $\{P^m \mathbb{1}_A > 0\} \subseteq \{\mathbb{E}(\mathbb{1}_A | \Sigma_{\text{inv}}) > 0\}$ up to a nullset.*

Proof. Let $A \in \Sigma$ and fix a Σ_{inv} -measurable representative of $\mathbb{E}(\mathbb{1}_A | \Sigma_{\text{inv}})$. Then for $B := \{\mathbb{E}(\mathbb{1}_A | \Sigma_{\text{inv}}) = 0\} \in \Sigma_{\text{inv}}$ we have

$$\begin{aligned} 0 &= \int_B \mathbb{E}(\mathbb{1}_A | \Sigma_{\text{inv}}) \, d\mu \leftarrow \int_B \frac{1}{n} \sum_{m=0}^{n-1} P^m \mathbb{1}_A \, d\mu = \frac{1}{n} \sum_{m=0}^{n-1} \int_B P^m \mathbb{1}_A \, d\mu \\ &= \frac{1}{n} \sum_{m=0}^{n-1} \int_{\varphi^{-m}(B)} \mathbb{1}_A \, d\mu = \mu(A \cap B). \end{aligned}$$

Since the right-hand side does not depend on $n \in \mathbb{N}_0$, it follows that $\int_B P^m \mathbb{1}_A \, d\mu = 0$ for all $m \in \mathbb{N}_0$ and therefore $\{P^m \mathbb{1}_A > 0\} \subseteq \{\mathbb{E}(\mathbb{1}_A | \Sigma_{\text{inv}}) > 0\}$ up to a nullset. \square

Remark 3.3. Let us briefly recall the well-known fact that for a measure preserving dynamical system $(\Omega, \Sigma, \mu; \varphi)$ we have

$$\overline{\Sigma_{\text{inv}}} = \{A \in \Sigma : A \sim \varphi^{-1}(A)\}.$$

Indeed, every $A \in \overline{\Sigma_{\text{inv}}}$ coincides with some set in Σ_{inv} up to a nullset and hence satisfies $\varphi^{-1}(A) \sim A$. Conversely, given a set $A \in \Sigma$ such that $A \sim \varphi^{-1}(A)$, it can easily be checked that

$$A \sim \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} \varphi^{-k}(A) \in \Sigma_{\text{inv}},$$

which implies that $A \in \overline{\Sigma_{\text{inv}}}$.

As mentioned in the introduction, we always have $\Sigma_{\text{inv}} \subseteq \Sigma_\infty$, whereas the converse inclusion does not hold in general. In fact, $\Sigma_\infty = \Sigma$ whenever φ is bimeasurable and bijective. The next lemma gives a sufficient condition for $\overline{\Sigma_{\text{inv}}} = \Sigma_\infty$ to hold.

Lemma 3.4. *Let $(\Omega, \Sigma, \mu; \varphi)$ be a bimeasurable measure preserving dynamical system such that the sequence $(\varphi^n(A))$ converges in Σ/\sim . Then $\Sigma_\infty \subseteq \overline{\Sigma_{\text{inv}}}$.*

Proof. Let $B \in \Sigma_\infty$. Then for every $n \in \mathbb{N}$ there exists $A_n \in \Sigma$ such that $B = \varphi^{-n}(A_n)$. Consequently, $\varphi^n(B) \subseteq A_n$ for all $n \in \mathbb{N}$. Since $B \subseteq \varphi^{-n}(\varphi^n(B))$ we have

$$\mu(\varphi^n(B)) = \mu(\varphi^{-n}(\varphi^n(B))) \geq \mu(B) = \mu(\varphi^{-n}(A_n)) = \mu(A_n)$$

for all $n \in \mathbb{N}$. This shows that $\varphi^n(B) \sim A_n$ for each $n \in \mathbb{N}$. Hence, by assumption, $\lim A_n =: A$ exists in Σ/\sim . Since $d(\varphi^{-1}(C), \varphi^{-1}(D)) = d(C, D)$ for all $C, D \in \Sigma$, the mapping $\varphi^{-1}: \Sigma/\sim \rightarrow \Sigma/\sim$ is an isometry and in particular continuous. Hence,

$$\begin{aligned} d(\varphi^{-1}(B), B) &= d(\varphi^{-(n+1)}(A_n), \varphi^{-n}(A_n)) \\ &\leq d(\varphi^{-(n+1)}(A_n), \varphi^{-(n+1)}(A_{n+1})) + d(\varphi^{-(n+1)}(A_{n+1}), \varphi^{-n}(A_n)) \\ &= d(A_n, A_{n+1}) + d(B, B) \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

This shows that $B \sim \varphi^{-1}(B)$ and thus, by Remark 3.3, $B \in \overline{\Sigma_{\text{inv}}}$. \square

The next lemma will be used to identify the limit of the sequence $(\varphi^n(A))_{n \in \mathbb{N}}$ in Σ/\sim . Beside this, it shows measurability of minimal invariant supersets.

Lemma 3.5. *Let $(\Omega, \Sigma, \mu; \varphi)$ be a bimeasurable measure preserving dynamical system. Then the minimal invariant superset of any $A \in \Sigma$ is itself measurable, i.e. $A^* \in \Sigma_{\text{inv}}$, and is given by*

$$A^* = \bigcup_{m \in \mathbb{N}_0} \varphi^{-m} \left(\bigcup_{n \in \mathbb{N}_0} \varphi^n(A) \right) \sim \bigcup_{n \in \mathbb{N}_0} \varphi^n(A).$$

Proof. Let $A \in \Sigma$ and set

$$B := \bigcup_{n \in \mathbb{N}_0} \varphi^n(A).$$

Then $A \subseteq B$ and obviously $\varphi(B) \subseteq B$. Hence, the sequence $(\varphi^{-n}(B))_{n \in \mathbb{N}_0}$ is increasing and therefore,

$$D := \bigcup_{n \in \mathbb{N}_0} \varphi^{-n}(B)$$

is φ -invariant, i.e. $\varphi^{-1}(D) = D$. Clearly, $A \subseteq D$ and $D \in \Sigma_{\text{inv}}$.

Now let $A \subseteq F \subseteq \Omega$ such that $\varphi^{-1}(F) = F$. Since $\varphi^n(A) \subseteq \varphi^n(F) = \varphi^n(\varphi^{-n}(F)) \subseteq F$ for every $n \in \mathbb{N}_0$, we know that $B \subseteq F$. Therefore, $\varphi^{-n}(B) \subseteq \varphi^{-n}(F) = F$ for every $n \in \mathbb{N}_0$, which implies that $D \subseteq F$. Since F was arbitrary, this shows that D is in fact the minimal invariant superset of A , denoted by A^* .

Finally, we may conclude from the measure preservation that $\varphi^{-1}(B) \sim B$ which implies that $D \sim B$. \square

One implication of Theorem 1.1 is based on the following results by Lin from [10, Thm 4.3]; see also [2, Thm 2] for a generalization.

Theorem 3.6. *Let X be a Banach space and $P \in \mathcal{L}(X)$ be a contractive linear operator. We denote by $T \in \mathcal{L}(X^*)$ the adjoint of P and set $B_{X^*} := \{g \in X^* : \|g\| \leq 1\}$. Then for any $f \in X$ the following are equivalent:*

- (i) $\lim P^n f = 0$.
- (ii) $\langle f, h \rangle = 0$ for all $h \in \mathcal{F} := \bigcap_{n \in \mathbb{N}} T^n B_{X^*}$.

Now we turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. (i) \Rightarrow (iii): This implication follows from Lemma 3.4 and the abovementioned remark.

(iii) \Rightarrow (ii): Assume that $\Sigma_\infty \subseteq \overline{\Sigma_{\text{inv}}}$. Let $f \in L^1(\Omega, \Sigma, \mu)$ and define $\tilde{f} := f - \mathbb{E}(f|\Sigma_{\text{inv}})$. By Remark 3.1 we obtain that $\lim P^n f = \mathbb{E}(f|\Sigma_{\text{inv}})$ if and only if $\lim P^n \tilde{f} = 0$. In view of Theorem 3.6 it thus suffices to show that $\langle \tilde{f}, h \rangle = 0$ for each $h \in \bigcap_{n \in \mathbb{N}} T^n B_{L^\infty}$.

So let $h \in \bigcap_{n \in \mathbb{N}} T^n B_{L^\infty}$. Then for each $n \in \mathbb{N}$ there exists $g_n \in B_{L^\infty}$ such that $h = g_n \circ \varphi^n$ in the sense of L^∞ . Let us fix Σ -measurable representatives of h and all g_n and let $A \in \Sigma$. Then for every $n \in \mathbb{N}$ we have $h^{-1}(A) \sim \varphi^{-n}(g_n^{-1}(A)) \in \Sigma_n$. Since $\Sigma_\infty \subseteq \overline{\Sigma_{\text{inv}}}$ we obtain that

$$h^{-1}(A) \sim \bigcap_{n \in \mathbb{N}} \varphi^{-n}(g_n^{-1}(A)) \in \Sigma_\infty \subseteq \overline{\Sigma_{\text{inv}}}.$$

Due to the completeness of $\overline{\Sigma_{\text{inv}}}$, this already implies that $h^{-1}(A) \in \overline{\Sigma_{\text{inv}}}$. We hence showed that any $h \in \bigcap_{n \in \mathbb{N}} T^n B_{L^\infty}$ is $\overline{\Sigma_{\text{inv}}}$ -measurable.

By definition of the conditional expectation we know that

$$\langle \tilde{f}, \mathbb{1}_B \rangle = \int_B f \, d\mu - \int_B \mathbb{E}(f|\Sigma_{\text{inv}}) \, d\mu = 0$$

for all $B \in \overline{\Sigma_{\text{inv}}}$. By linearity and density it thus follows that $\langle \tilde{f}, h \rangle = 0$ for all $h \in \bigcap_{n \in \mathbb{N}} T^n B_{L^\infty}$. As said in the beginning, assertion (ii) now follows from Theorem 3.6.

(ii) \Rightarrow (i): Assume that $(P^n)_{n \in \mathbb{N}}$ is strongly convergent and let $A \in \Sigma$. We fix Σ -measurable representatives f_n of $P^n \mathbb{1}_A$ and a Σ_{inv} -measurable representative g of $\mathbb{E}(\mathbb{1}_A|\Sigma_{\text{inv}})$. Let $S := \text{supp } g := \{g > 0\}$. For a given $\varepsilon > 0$ we may choose a constant $c > 0$ small enough such that $\mu(S \cap \{g < c\}) < \varepsilon$. By assumption and Remark 3.1 we find $N \in \mathbb{N}$ such that $\|f_n - g\|_{L^1} < \varepsilon \cdot c$ for all $n \geq N$. Since $\{f_n > 0\} \subseteq \{g > 0\}$ up to a nullset by Lemma 3.2 and

$$c \cdot \mu(\{f_n = 0\} \cap \{g \geq c\}) \leq \|f_n - g\|_{L^1} < \varepsilon \cdot c$$

for all $n \geq N$, it follows that

$$\begin{aligned} d(\{f_n > 0\}, \{g > 0\}) &= \mu(S \setminus \{f_n > 0\}) \\ &= \mu(S \cap \{g < c\} \setminus \{f_n > 0\}) + \mu(S \cap \{g \geq c\} \setminus \{f_n > 0\}) \leq 2\varepsilon \end{aligned}$$

for all $n \geq N$. This shows that $\text{supp } P^n \mathbb{1}_A$ converges to $\text{supp } \mathbb{E}(\mathbb{1}_A|\Sigma_{\text{inv}})$ in Σ/\sim . Furthermore, since P is a Markov operator and $A \subseteq \varphi^{-n}(\varphi^n(A))$, we have

$$\mu(A) = \int_\Omega P^n \mathbb{1}_A \, d\mu \geq \int_{\varphi^n(A)} P^n \mathbb{1}_A \, d\mu = \int_{\varphi^{-n}(\varphi^n(A))} \mathbb{1}_A \, d\mu \geq \int_A \mathbb{1}_A \, d\mu = \mu(A)$$

for all $n \in \mathbb{N}_0$; this implies that $\text{supp } P^n \mathbb{1}_A \subseteq \varphi^n(A)$ up to a nullset.

In summary, we proved that $\varphi^n(A)$ is (up to a nullset) a superset of the support of $P^n \mathbb{1}_A$, which converges to the support of $\mathbb{E}(\mathbb{1}_A|\Sigma_{\text{inv}})$ in Σ/\sim . By Lemma 3.2, we also know that A , which is nothing but the support of $P^0 \mathbb{1}_A$, is contained in the support of $\mathbb{E}(\mathbb{1}_A|\Sigma_{\text{inv}})$ up to a nullset. Applying this observation to each of the sets $\varphi^m(A)$, we conclude that for any $m \in \mathbb{N}_0$ and $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $\mu(\varphi^m(A) \setminus \varphi^n(\varphi^m(A))) < \varepsilon$ whenever $n \geq k$. Now we show that the sequence $(\varphi^n(A))_{n \in \mathbb{N}}$ converges to

$$U := \bigcup_{n \in \mathbb{N}_0} \varphi^n(A)$$

in Σ/\sim . Fix $\varepsilon > 0$ and $N \in \mathbb{N}$ such that

$$\mu\left(U \setminus \bigcup_{0 \leq m \leq N} \varphi^m(A)\right) < \varepsilon.$$

By what we have just seen, there exists $k \in \mathbb{N}$ such that $\mu(\varphi^m(A) \setminus \varphi^{n+m}(A)) < \varepsilon/N$ for every $n \geq k$ and all $1 \leq m \leq N$. Therefore,

$$\mu(U \setminus \varphi^n(A)) \leq \mu\left(\bigcup_{1 \leq m \leq N} \varphi^m(A) \setminus \varphi^n(A)\right) + \varepsilon \leq \varepsilon + \sum_{m=1}^N \mu(\varphi^m(A) \setminus \varphi^n(A)) \leq 2\varepsilon$$

for all $n \geq k + N$. Since clearly $\varphi^n(A) \subseteq U$ for every $n \in \mathbb{N}$, this proves that $\lim \varphi^n(A) = U$ in Σ/\sim as claimed. It now follows from Lemma 3.5 that

$$U \sim \bigcup_{n \in \mathbb{N}_0} \varphi^{-n}(U) = A^*,$$

which completes the proof. \square

REFERENCES

- [1] V. I. Bogachev. *Measure theory. Vol. I, II.* Springer-Verlag, Berlin, 2007.
- [2] Y. Derriennic. Lois “zéro ou deux” pour les processus de Markov. Applications aux marches aléatoires. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 12(2):111–129, 1976.
- [3] J. Ding and A. Zhou. *Statistical properties of deterministic systems.* Tsinghua University Texts. Springer-Verlag, Berlin; Tsinghua University Press, Beijing, 2009.
- [4] Y. Ding. The asymptotic behavior of Frobenius-Perron operator with local lower-bound function. *Chaos Solitons Fractals*, 18(2):311–319, 2003.
- [5] T. Eisner, B. Farkas, M. Haase, and R. Nagel. *Operator Theoretic Aspects of Ergodic Theory*, volume 272 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2015.
- [6] S. R. Foguel. *The ergodic theory of Markov processes.* Van Nostrand Mathematical Studies, No. 21. Van Nostrand Reinhold Co., New York-Toronto, Ont.-London, 1969.
- [7] M. Gerlach and J. Glück. Lower Bounds and the Asymptotic Behaviour of Positive Operator Semigroups. *Ergod. Th. & Dynam. Sys.*, to appear.
- [8] A. Lasota. Statistical stability of deterministic systems. In *Equadiff 82 (Würzburg, 1982)*, volume 1017 of *Lecture Notes in Math.*, pages 386–419. Springer, Berlin, 1983.
- [9] A. Lasota and J. A. Yorke. Exact dynamical systems and the Frobenius-Perron operator. *Trans. Amer. Math. Soc.*, 273(1):375–384, 1982.
- [10] M. Lin. Mixing for Markov operators. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 19:231–242, 1971.
- [11] R. D. Mauldin. Bimeasurable functions. *Proc. Amer. Math. Soc.*, 83(2):369–370, 1981.
- [12] R. Purves. Bimeasurable functions. *Fund. Math.*, 58:149–157, 1966.
- [13] R. E. Rice. On mixing transformations. *Aequationes Math.*, 17(1):104–108, 1978.

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